

Debt Sustainability Condition – Mathematical Version

The famous debt sustainability condition is often written mathematically as follows:

$$\dot{L} = \frac{n}{L} L - F; \quad \dot{E} = \frac{n}{E} E - F$$

SIMPLE ANALYTICS:

Public Debt-to-GDP ratio will rise if:

$$\frac{\dot{L}}{L} > \frac{\dot{E}}{E}$$

Debt sustainability requires:

$$\frac{\dot{L}}{L} \leq \frac{\dot{E}}{E}$$

If debt monetization is pursued ($\dot{L} = P$), it will help reduce the debt burden. However, debt monetization may result in high inflation. If we rule out debt monetization ($\dot{L} = L$), then debt sustainability requires

$$F \leq \frac{s}{L} L - F; \quad \dot{E} = \frac{n}{E} E - F$$

or, $\dot{L} = L$ implies $\dot{L} = L$ and $\dot{E} = \frac{n}{E} E - F$

$$\frac{\dot{L}}{L} = 1; \quad \frac{\dot{E}}{E} = \frac{n}{E} - \frac{F}{E}$$

\dot{L} : Government Purchases

\dot{E} : Taxes net of Transfers ($\dot{E} = T - G = J - A - N - O$)

$\dot{L} - \dot{E}$: Primary Budget Deficit (note: $\dot{L} = P$ implies a primary budget deficit)

L : Government Debt in Period t

DERIVATION

Reference: Macroeconomics (2nd Edition) by R. Glenn Hubbard and Anthony Patrick O'Brien
(Publisher: Pearson)

Standard budget deficit is given by:

$$G_t - T_t = \Delta D_t + \Delta M_t$$

To capture the role of seigniorage (essentially represents a transfer of wealth from individuals holding money to the government), we extend the above equation to include change in monetary base:

$$G_t - T_t = \Delta D_t + \Delta M_t + \Delta MB_t$$

Note that the change in government debt is given by:

$$\Delta D_t = (1+r)D_{t-1} - N_t + S_t$$

Divide both sides by nominal GDP and rearrange to get

$$\frac{G_t - T_t}{Y_t} = \frac{(1+r)D_{t-1}}{Y_t} - \frac{N_t}{Y_t} + \frac{S_t}{Y_t} + \frac{\Delta MB_t}{Y_t}$$

Modify previous equation as follows (multiply and divide the first term on the right hand side by $\frac{Y_{t-1}}{Y_t}$):

$$\frac{G_t - T_t}{Y_t} = \frac{(1+r)D_{t-1}}{Y_{t-1}} \frac{Y_{t-1}}{Y_t} - \frac{N_t}{Y_t} + \frac{S_t}{Y_t} + \frac{\Delta MB_t}{Y_t}$$

Note the following condition,

$$\frac{Y_{t-1}}{Y_t} \approx 1 - \pi_t$$

$$\frac{(1+r)D_{t-1}}{Y_{t-1}} \approx (1+r) \frac{D_{t-1}}{Y_{t-1}} - \pi_t \frac{D_{t-1}}{Y_{t-1}}$$

Using the approximation

$$(1+r) \frac{D_{t-1}}{Y_{t-1}} \approx (1+r) \frac{D_{t-1}}{Y_{t-1}} - \pi_t \frac{D_{t-1}}{Y_{t-1}}$$

We get:

$$\frac{G_t - T_t}{Y_t} = (1+r) \frac{D_{t-1}}{Y_{t-1}} - \pi_t \frac{D_{t-1}}{Y_{t-1}} - \frac{N_t}{Y_t} + \frac{S_t}{Y_t} + \frac{\Delta MB_t}{Y_t}$$

Another useful approximation:

$$I = \frac{U E_{\dot{s}}}{U E_{\dot{s}} E_{\dot{s}}} p N U E_{\dot{s}} F \hat{E}_{\dot{s}} F_{\dot{s}}$$

So

$$\frac{n_{\dot{s}}}{|\dot{s} \cdot \dot{s}} L : U E_{\dot{s}} F \hat{E}_{\dot{s}} F_{\dot{s}}; \quad I = \frac{n_{\dot{s}} \dot{U}}{|\dot{s} \dot{U} \cdot \dot{s} \dot{U}} p E \frac{s_{\dot{s}}}{|\dot{s} \cdot \dot{s}} F \frac{\epsilon_{\dot{s}}}{|\dot{s} \cdot \dot{s}} F \frac{\dot{t}_{\dot{s}}}{|\dot{s} \cdot \dot{s}}$$

Rearrange terms to get: